NOTE ON 2D SCHRÖDINGER OPERATORS WITH δ-INTERACTIONS ON ANGLES AND CROSSING LINES

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Abstract. In this note we sharpen the lower bound from [LLP10] on the spectrum of the 2D Schrödinger operator with a δ-interaction supported on a planar angle. Using the same method we obtain the lower bound on the spectrum of the 2D Schrödinger operator with a δ-interaction supported on crossing straight lines. The latter operators arise in the three-body quantum problem with δ-interactions between particles.

1. Introduction

Self-adjoint Schrödinger operators with δ-interactions supported on sufficiently regular hypersurfaces can be defined via closed, densely defined, symmetric and lower-semibounded quadratic forms using the first representation theorem, see [BEKS94] and also [BLL13].

δ-interactions on angles. In our first model the support of the δ-interaction is the set $\Sigma_\varphi \subset \mathbb{R}^2$, which consists of two rays meeting at the common origin and constituting the angle $\varphi \in (0, \pi]$ as in Figure 1.

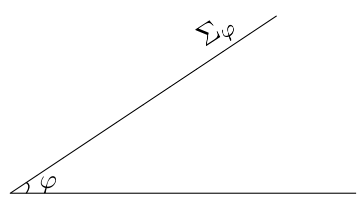

Figure 1. The angle Σ_φ of degree $\varphi \in (0, \pi]$.

The quadratic form in $L^2(\mathbb{R}^2)$

$$\mathfrak{a}_\varphi[f] := \|\nabla f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} - \alpha \|f|_{\Sigma_\varphi}\|^2_{L^2(\Sigma_\varphi)}, \qquad \operatorname{dom} \mathfrak{a}_\varphi := H^1(\mathbb{R}^2), \tag{1.1}$$

is closed, densely defined, symmetric and lower-semibounded, where $f|_{\Sigma_\varphi}$ is the trace of f on Σ_φ, and the constant $\alpha > 0$ is called the strength of interaction. The corresponding self-adjoint operator in $L^2(\mathbb{R}^2)$ we denote by A_φ. Known spectral properties of this operator include explicit representation of the essential spectrum $\sigma_{\mathrm{ess}}(A_\varphi) = [-\alpha^2/4, +\infty)$ and some information on the discrete spectrum: $\sharp\sigma_{\mathrm{d}}(A_\varphi) \geq 1$ if and only if $\varphi \neq \pi$. These two statements can be deduced from more general results by Exner and Ichinose [EI01]. They are complemented by Exner and Nemčová in [EN03] with the limiting property $\sharp\sigma_{\mathrm{d}}(A_\varphi) \to +\infty$ as $\varphi \to 0+$.

In [LLP10] the author obtained jointly with Igor Lobanov and Igor Yu. Popov a general result, which implies the lower bound on the spectrum of A_φ

$$\inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{4\sin^2(\varphi/2)}. \tag{1.2}$$

This bound is close to optimal for φ close to π, whereas in the limit $\varphi \to 0+$ the bound tends to $-\infty$. In the present note we sharpen this bound. Namely, we obtain

$$\inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{(1+\sin(\varphi/2))^2}. \tag{1.3}$$

The new bound yields that the operators A_φ are uniformly lower-semibounded with respect to φ and

$$\inf \sigma(A_\varphi) \geq -\alpha^2$$

holds for all $\varphi \in (0, \pi]$. This observation agrees well with physical expectations. Note that separation of variables yields that $\inf \sigma(A_\pi) = -\alpha^2/4$ and in this case the lower bound in (1.3) coincides with the exact spectral bottom.

For sufficiently sharp angles upper bounds on $\inf \sigma(A_\varphi)$ were obtained by Brown, Eastham and Wood in [BEW08]. See also Open Problem 7.3 in [E08] related to the discrete spectrum of A_φ for φ close to π.

δ-interactions on crossing straight lines. We also consider an analogous model with the δ-interaction supported on the set $\Gamma_\varphi = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two straight lines, which cross at the angle $\varphi \in (0, \pi)$ as in Figure 2.

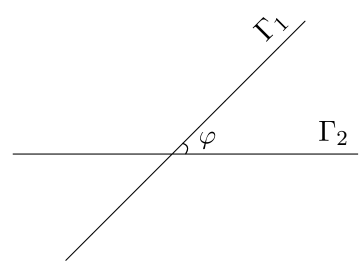

FIGURE 2. The straight lines Γ_1 and Γ_2 crossing at the angle of degree $\varphi \in (0, \pi)$.

The corresponding self-adjoint operator B_φ in $L^2(\mathbb{R}^2)$ can be defined via the closed, densely defined, symmetric and lower-semibounded quadratic form

$$\mathfrak{b}_\varphi[f] := \|\nabla f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} - \alpha\|f|_{\Gamma_\varphi}\|^2_{L^2(\Gamma_\varphi)}, \qquad \operatorname{dom}\mathfrak{b}_\varphi := H^1(\mathbb{R}^2), \tag{1.4}$$

in $L^2(\mathbb{R}^2)$, where $\alpha > 0$ is the strength of interaction. According to [EN03] it is known that $\sigma_{\rm ess}(B_\varphi) = [-\alpha^2/4, +\infty)$ and that $\sharp\sigma_{\rm d}(B_\varphi) \geq 1$.

In this note we obtain the lower bound

$$\inf\sigma(B_\varphi) \geq -\frac{\alpha^2}{1+\sin\varphi}, \tag{1.5}$$

using the same method as for the operator A_φ. Separation of variables yields $\inf\sigma(B_{\pi/2}) = -\alpha^2/2$, and in this case the lower bound in the estimate (1.5) coincides with the exact spectral bottom.

Upper bounds on $\inf\sigma(B_\varphi)$ were obtained in [BEW08, BEW09]. The operators of the type B_φ arise in the one-dimensional quantum three-body problem after excluding the center of mass, see Cornean, Duclos and Ricaud [CDR06, CDR08] and the references therein.

We want to stress that our proofs are of elementary nature and we do not use any reduction to integral operators acting on interaction supports Σ_φ and Γ_φ.

2. Sobolev spaces on wedges

In this section $\Omega \subset \mathbb{R}^2$ is a wedge with the angle of degree $\varphi \in (0, 2\pi)$. The Sobolev space $H^1(\Omega)$ is defined as usual, see [McL, Chapter 3]. For any $f \in H^1(\Omega)$ the trace $f|_{\partial\Omega} \in L^2(\partial\Omega)$ is well-defined as in [McL, Chapter 3] and [M87].

Proposition 2.1. [LP08, Lemma 2.6] *Let Ω be a wedge with angle of degree $\varphi \in (0,\pi]$. Then for any $f \in H^1(\Omega)$ the estimate*

$$\|\nabla f\|^2_{L^2(\Omega;\mathbb{C}^2)} - \gamma\|f|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} \geq -\frac{\gamma^2}{\sin^2(\varphi/2)}\|f\|^2_{L^2(\Omega)}$$

holds for all $\gamma > 0$.

Proposition 2.2. [LP08, Lemma 2.8] *Let Ω be a wedge with angle of degree $\varphi \in (\pi, 2\pi)$. Then for any $f \in H^1(\Omega)$ the estimate*

$$\|\nabla f\|^2_{L^2(\Omega;\mathbb{C}^2)} - \gamma\|f|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} \geq -\gamma^2\|f\|^2_{L^2(\Omega)}$$

holds for all $\gamma > 0$.

Propositions 2.1 and 2.2 are variational equivalents of spectral results from [LP08].

3. A lower bound on the spectrum of A_φ

In the next theorem we sharpen the bound (1.2) using only properties of the Sobolev space H^1 on wedges and some optimization.

Theorem 3.1. *Let the self-adjoint operator A_φ be associated with the quadratic form given in (1.1). Then the estimate*

$$\inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{\big(1+\sin(\varphi/2)\big)^2}$$

holds.

Proof. The angle Σ_φ separates the Euclidean space $\mathbb{R}^2$ into two wedges Ω_1 and Ω_2 with angles of degrees φ and $2\pi-\varphi$, see Figure 3.

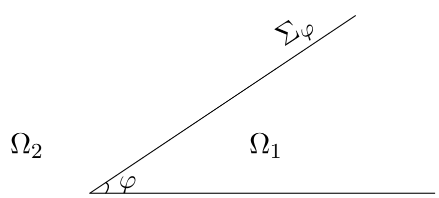

Figure 3. The angle Σ_φ separates the Euclidean space $\mathbb{R}^2$ into two wedges Ω_1 and Ω_2.

The underlying Hilbert space can be decomposed as

$$L^2(\mathbb{R}^2) = L^2(\Omega_1) \oplus L^2(\Omega_2).$$

Any $f \in \operatorname{dom} \mathfrak{a}_\varphi$ can be written as the orthogonal sum $f_1 \oplus f_2$ with respect to that decomposition of $L^2(\mathbb{R}^2)$. Note that $f_1 \in H^1(\Omega_1)$ and that $f_2 \in H^1(\Omega_2)$. Clearly,

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^2)} &= \|f_1\|^2_{L^2(\Omega_1)} + \|f_2\|^2_{L^2(\Omega_2)}, \\ \|\nabla f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} &= \|\nabla f_1\|^2_{L^2(\Omega_1;\mathbb{C}^2)} + \|\nabla f_2\|^2_{L^2(\Omega_2;\mathbb{C}^2)}. \end{aligned} \tag{3.1}$$

The coupling constant can be decomposed as $\alpha = \beta + (\alpha-\beta)$ with some optimization parameter $\beta \in [0,\alpha]$ and the relation

$$\alpha\|f|_{\Sigma_\varphi}\|^2_{L^2(\Sigma_\varphi)} = \beta\|f_1|_{\partial\Omega_1}\|^2_{L^2(\partial\Omega_1)} + (\alpha-\beta)\|f_2|_{\partial\Omega_2}\|^2_{L^2(\partial\Omega_2)}. \tag{3.2}$$

holds. According to Proposition 2.1

$$\|\nabla f_1\|^2_{L^2(\Omega_1;\mathbb{C}^2)} - \beta\|f_1|_{\partial\Omega_1}\|^2_{L^2(\partial\Omega_1)} \geq -\frac{\beta^2}{\sin^2(\varphi/2)}\|f_1\|^2_{L^2(\Omega_1)}, \tag{3.3}$$

and according to Proposition 2.2

$$\|\nabla f_2\|^2_{L^2(\Omega_2;\mathbb{C}^2)} - (\alpha-\beta)\|f_2|_{\partial\Omega_2}\|^2_{L^2(\partial\Omega_2)} \geq -(\alpha-\beta)^2\|f_2\|^2_{L^2(\Omega_2)}. \tag{3.4}$$

The observations (3.1), (3.2) and the estimates (3.3), (3.4) imply

$$\mathfrak{a}_\varphi[f] \geq -\max\left\{\frac{\beta^2}{\sin^2(\varphi/2)}, (\alpha-\beta)^2\right\}\|f\|^2_{L^2(\mathbb{R}^2)}.$$

Making optimization with respect to β, we observe that the maximum between the two values in the estimate above is minimal, when these two values coincide. That is

$$\frac{\beta^2}{\sin^2(\varphi/2)} = (\alpha-\beta)^2,$$

which is equivalent to

$$\beta = \frac{\alpha\sin(\varphi/2)}{(1+\sin(\varphi/2))}, \tag{3.5}$$

resulting in the final estimate

$$\mathfrak{a}_\varphi[f] \geq -\frac{\alpha^2}{(1+\sin(\varphi/2))^2}\|f\|^2_{L^2(\mathbb{R}^2)}.$$

This final estimate implies the desired spectral bound. □

Remark 3.2. Note that the previously known lower bound (1.2) comes out from the proof of the last theorem if we choose $\beta = \alpha/2$, which is the optimal choice in our proof only for $\varphi = \pi$ as we see from (3.5).

4. A lower bound on the spectrum of B_φ

In the next theorem we obtain a lower bound on the spectrum of the self-adjoint operator B_φ using the same idea as in Theorem 3.1.

Theorem 4.1. *Let the self-adjoint operator B_φ be associated with the quadratic form given in* (1.4). *Then the estimate*

$$\inf\sigma(B_\varphi) \geq -\frac{\alpha^2}{1+\sin\varphi}$$

holds.

Proof. The crossing straight lines Γ_1 and Γ_2 separate the Euclidean space $\mathbb{R}^2$ into four wedges $\{\Omega_k\}_{k=1}^4$. Namely, the wedges Ω_1 and Ω_2 with angles of degree φ and the wedges Ω_3 and Ω_4 with angles of degree $\pi-\varphi$, see Figure 4.

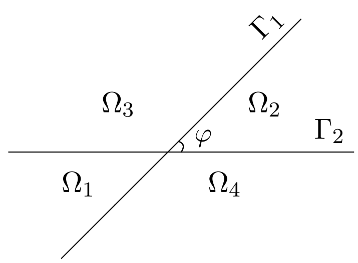

FIGURE 4. The crossing straight lines Γ_1 and Γ_2 separate the Euclidean space $\mathbb{R}^2$ into four wedges $\{\Omega_k\}_{k=1}^4$.

The underlying Hilbert space can be decomposed as

$$L^2(\mathbb{R}^2) = \bigoplus_{k=1}^{4} L^2(\Omega_k).$$

Any $f \in \operatorname{dom} \mathfrak{b}_\varphi$ can be written as the orthogonal sum $\oplus_{k=1}^4 f_k$ with respect to that decomposition of $L^2(\mathbb{R}^2)$. Note that $f_k \in H^1(\Omega_k)$ for $k = 1, 2, 3, 4$. Clearly,

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \sum_{k=1}^{4} \|f_k\|^2_{L^2(\Omega_k)}, \quad \|\nabla f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} = \sum_{k=1}^{4} \|\nabla f_k\|^2_{L^2(\Omega_k;\mathbb{C}^2)}. \tag{4.1}$$

The coupling constant can be decomposed as $\alpha = \beta + (\alpha - \beta)$ with some optimization parameter $\beta \in [0, \alpha]$ and the relation

$$\begin{aligned} \alpha \|f|_{\Gamma_\varphi}\|^2_{L^2(\Gamma_\varphi)} &= \beta\|f_1|_{\partial\Omega_1}\|^2_{L^2(\partial\Omega_1)} + \beta\|f_2|_{\partial\Omega_2}\|^2_{L^2(\partial\Omega_2)} \\ &\quad + (\alpha-\beta)\|f_3|_{\partial\Omega_3}\|^2_{L^2(\partial\Omega_3)} + (\alpha-\beta)\|f_4|_{\partial\Omega_4}\|^2_{L^2(\partial\Omega_4)} \end{aligned} \tag{4.2}$$

holds. According to Proposition 2.1

$$\begin{aligned} \|\nabla f_1\|^2_{L^2(\Omega_1;\mathbb{C}^2)} - \beta\|f_1|_{\partial\Omega_1}\|^2_{L^2(\partial\Omega_1)} &\geq -\tfrac{\beta^2}{\sin^2(\varphi/2)}\|f_1\|^2_{L^2(\Omega_1)}, \\ \|\nabla f_2\|^2_{L^2(\Omega_2;\mathbb{C}^2)} - \beta\|f_2|_{\partial\Omega_2}\|^2_{L^2(\partial\Omega_2)} &\geq -\tfrac{\beta^2}{\sin^2(\varphi/2)}\|f_2\|^2_{L^2(\Omega_2)}. \end{aligned} \tag{4.3}$$

Also according to Proposition 2.1

$$\begin{aligned} \|\nabla f_3\|^2_{L^2(\Omega_3;\mathbb{C}^2)} - (\alpha-\beta)\|f_3|_{\partial\Omega_3}\|^2_{L^2(\partial\Omega_3)} &\geq -\tfrac{(\alpha-\beta)^2}{\cos^2(\varphi/2)}\|f_3\|^2_{L^2(\Omega_3)}, \\ \|\nabla f_4\|^2_{L^2(\Omega_4;\mathbb{C}^2)} - (\alpha-\beta)\|f_4|_{\partial\Omega_4}\|^2_{L^2(\partial\Omega_4)} &\geq -\tfrac{(\alpha-\beta)^2}{\cos^2(\varphi/2)}\|f_4\|^2_{L^2(\Omega_4)}. \end{aligned} \tag{4.4}$$

The observations (4.1), (4.2) and the estimates (4.3), (4.4) imply

$$\mathfrak{b}_\varphi[f] \geq -\max\left\{\tfrac{\beta^2}{\sin^2(\varphi/2)}, \tfrac{(\alpha-\beta)^2}{\cos^2(\varphi/2)}\right\}\|f\|^2_{L^2(\mathbb{R}^2)}.$$

Making optimization with respect to β, we observe that the maximum between the two values in the estimate above is minimal, when these two values coincide. That is

$$\tfrac{\beta^2}{\sin^2(\varphi/2)} = \tfrac{(\alpha-\beta)^2}{\cos^2(\varphi/2)},$$

which is equivalent to

$$\beta = \tfrac{\alpha\tan(\varphi/2)}{(1+\tan(\varphi/2))}, \tag{4.5}$$

resulting in the final estimate

$$\mathfrak{b}_\varphi[f] \geq -\tfrac{\alpha^2}{1+\sin(\varphi)}\|f\|^2_{L^2(\mathbb{R}^2)}.$$

This final estimate implies the desired spectral bound. ☐

Remark 4.2. The result of Theorem 4.1 complements [CDR08, Theorem 4.6 (iv)], where the bound

$$\inf \sigma(B_\varphi) \geq -\alpha^2.$$

for all $\varphi \in (0, \pi)$ was obtained.

5. Acknowledgements

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